ON POLE AND RESIDUE EXTRACTION IN THE FREQUENCY DOMAIN

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ABSTRACT

A data analysis technique is presented for extracting pole and residue content of spectral data where phase data is not available or is not accurate. The technique is applied to a sample spectrum and results are obtained to demonstrate the accuracy and limitations of the procedure.

INTRODUCTION

Since the response of a second-order system in general can be represented by a pole residue expansion in the complex frequency domain, there are a number of applications for techniques that can be used to extract these poles and residues from frequency domain data. Those techniques currently available require both amplitude and phase data [1-4]. In many cases, phase data is difficult to obtain with acceptable accuracy. Moreover, in some cases, only amplitude data are available [5]. Consequently, a pole and residue extraction technique that requires only amplitude data is quite useful.

A technique is presented here that can be used to extract the poles and the complex absolute value of the residues from frequency domain data where only the amplitude data are available. In order to illustrate the procedure, frequency response data with six simple poles are considered. The accuracy and limitations of the technique

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are discussed.

ANALYSIS

Using the standard pole residue expansion, second-order system impulse response data often can be expressed

$$F(j\omega) = \sum_{m=1}^{N} \frac{A_m}{j\omega - s_m} + N(j\omega) \qquad (1)$$

where $\omega=2\pi f$ is the radian frequency, $s_m = \sigma_m + j\omega_m$ is the m^{th} pole, A_m is the residue of the m^{th} pole and $N(j\omega)$ represents the noise, if any, in the data. In general, the poles occur in complex conjugate doublets.

When the complex response data $F(j\omega)$ are known, a recently developed curve-fitting algorithm can be used to extract the poles and residues [1-3]. Of course the noise contribution does affect the results and limits the accuracy of the extracted poles and residues [4]. Techniques have been developed that can mitigate the noise level problem [6].

When only magnitude data are available some modification is in order. To that end consider

$$|F(j\omega)|^2 = F(j\omega)F^*(j\omega) \qquad (2)$$

It is apparent that $|F(j\omega)|^2$ has the poles of $F(j\omega)$ plus the poles of $F^*(j\omega)$. By inspection, if s_m is the m^{th} pole of $F(j\omega)$ then $-s_m^*$ is the corresponding pole of $F^*(j\omega)$. Consequently, the poles of $|F(j\omega)|^2$ will occur in quadruplets, i.e., s_m, s_m^*, $-s_m$, $-s_m^*$. Moreover, it can be shown via a partial fraction expansion that the residues of $|F(j\omega)|^2$ are

$$A_m^{(2)} = \frac{-1}{2\sigma_m} |A_m|^2 - \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{A_m A_n^*}{s_m + s_n^*} \qquad (3)$$

where the prime on the summation indicates that the n=m term is omitted. For poles with separation

$$\omega_m - \omega_n \gg \sigma_n, \sigma_m \tag{4}$$

and with comparable residues, (3) yields

$$A_m^{(2)} \simeq \frac{-1}{2\sigma_m} |A_m|^2 \tag{5}$$

When the curve fitting routine is applied to (2) the quadruplet poles (s_m, s_m^*, $-s_m$, $-s_m^*$) are obtained with residues $A_m^{(2)}$. And if (6) is satisfied and comparable residues occur, then (5) yields the complex absolute values of the residues of $F(j\omega)$,

$$|A_m| \simeq \sqrt{-2\sigma_m A_m^{(2)}} \tag{6}$$

Note from (5) that the imaginary part of $A_m^{(2)}$ is a measure of the accuracy of (5) and consequently a measure of the accuracy of (6). Typically the imaginary part of $A_m^{(2)}$ should be at most only one-tenth the real part.

If the residue of one of the poles is very much larger than the other residues, then (6) can not be used directly for the smaller residues. Therefore, these poles with dominant residues must be removed before the smaller residues can be obtained. In order to remove the m' pole quadruplet from the input data, (2) is modified as follows:

$$F'(j\omega) = |F(j\omega)|^2 (j\omega - s_{m'})(j\omega - s_{m'}^*)(j\omega + s_{m'})(j\omega + s_{m'}^*) \tag{7}$$

And the corresponding residues of $F'(j\omega)$ are

$$A_m^{(2)} \simeq \frac{-1}{2\sigma_m} |A_m|^2 (s_m - s_{m'})(s_m - s_{m'}^*)(s_m + s_{m'})(s_m + s_{m'}^*) \tag{8}$$

Then by using (7) and (8) the amplitudes of the residues in the original data can be obtained.

RESULTS

In order to verify the foregoing formulation, a simple pole expansion is considered.

$$F(j\omega) = \sum_{n=1}^{3} \frac{A_n}{j\omega - s_n} + \text{conj. pole contrib.} \qquad (10)$$

where $A_1=A_2=A_3=j/2$, $s_1 = (-.05+j1)2\pi$, $s_2 = (-.125+j5)2\pi$ and $s_3 = (-.1+j10)2\pi$. Applying the curve fitting procedure to (10) yields the results shown in Table 1. Then using only amplitude data (2), the data shown in Table 2 are obtained. Along with these data, (6) is applied to compute the amplitudes of the residues. For these two data sets it is concluded that the procedure proposed to extract poles and residue amplitudes from spectral amplitude data does work. However, there is some reduction in accuracy.

Because the presented technique for extracting pole and residue information from spectral amplitude data increases the number of poles by a factor of two, the effects of limited sample windows were investigated. Three windows were used to cover the sample spectrum. The widths of the windows were selected to include all of the spectral peak within the window. Results comparable to those in Table 2 were obtained. However, this is an important result in that it is shown that a very complex spectrum can be analyzed by considering individual spectral peaks. Moreover, it is possible to use the windows to exclude regions where the signal-to-noise ratios are small or even less than one.

The presented technique for pole and residue extraction has been applied to x-ray emission spectra for qualitative and quantitative analysis. Excellent results have been obtained [7].

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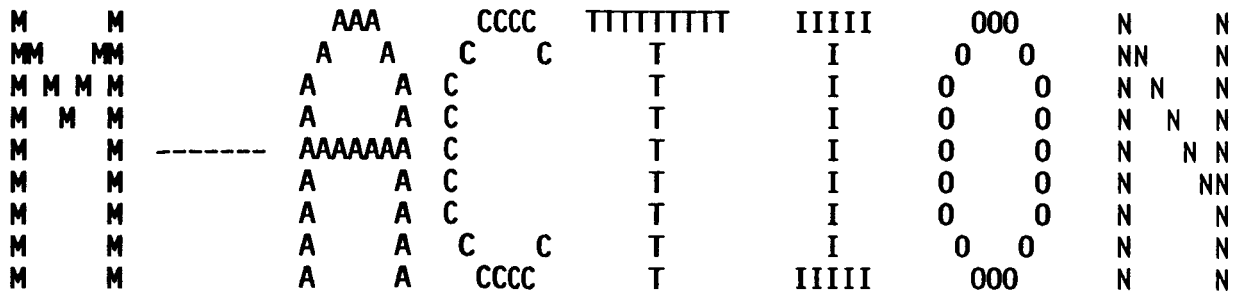
TABLE 1: Results from a least-square fit to the complex sample spectrum (10). For these data, 120 evenly spaced frequency samples were used with $f_{min} = 0.1$ and $f_{max} = 12$.

n	$S_n/2\pi$	A_n
1	-0.5000000 + j1.000000	$-.5701399 \times 10^{-14}$ + j0.5000000
2	-.12500000 + j5.000000	$.3399810 \times 10^{-14}$ + j0.5000000
3	-.10000000 + j10.00000	$.3553813 \times 10^{-15}$ + j0.5000000

TABLE 2: Results from a least-square fit to the square of the amplitude spectrum (10). For these data 120 evenly spaced frequency samples were used with $f_{min} = 0.1$ and $f_{max} = 12$.

n	$S_n/2\pi$	$A_n^{(2)}$	$\lvert A_n \rvert$
1	-.05000009 + j1.000001	.3971603 - j0.04443496	0.49957
2	-.12500000 + j5.000000	.1596142 - j0.01126768	0.50045
3	-.10000000 + j10.00000	.1992657 + j0.004103699	0.50045



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